HL Paper 3 Mock A 2021 – WORKED SOLUTIONS v2

1. [Maximum mark: 25]

(a) (i) $f(-3)=0,\ f(-1)=-1,\ f(1)=0,\ f(3)=1,\ f(5)=0$

$u(x)=x-3 \quad (u\circ f)(-3)=u(f(-3))=u(0)=-3$

$(u\circ f)(-1)=u(f(-1))=u(-1)=-4$

$(u\circ f)(1)=u(f(1))=u(0)=-3$

$(u\circ f)(3)=u(f(3))=u(1)=-2$

$(u\circ f)(5)=u(f(5))=u(0)=-3$

The above working is not necessary. It can be reasoned that since the range of f is $[-1,1]$ then the range of the composite function $u\circ f=u(f(x))$ will be $[-1-3,1-3]=[-4,-2]$.

(ii) $u\circ v\circ f=u(v(f(x)))$

Since $v(x)=2x$, then the range of $v\circ f=v(f(x))$ is $[2(-1),2(1)]=[-2,2]$

Thus, the range of $u\circ v\circ f$ will be $[-2-3,2-3]=[-5,-1]$

(iii) $f\circ v\circ u=f(v(u(x)))$

Since the domain of f is $[-3,5]$ then the range of $v\circ u=v(u(x))$ must be $[-3,5]$

$v(u(x))=v(x-3)=2(x-3)=2x-6$

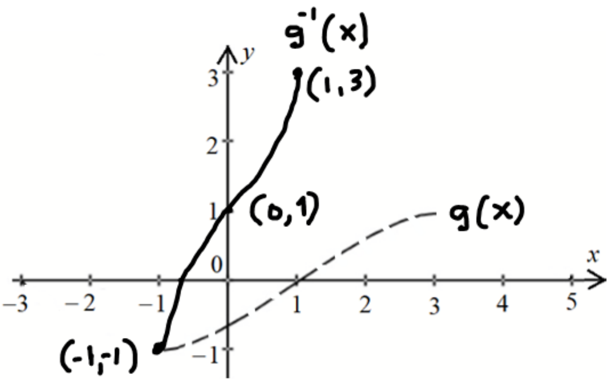
$2x-6=-3 \Rightarrow x=\frac{3}{2}$ and $2x-6=5 \Rightarrow x=\frac{11}{2}$

Thus, the largest possible domain for $f\circ v\circ u$ is $\left[\frac{3}{2},\frac{11}{2}\right]$.

(b) (i) f is not a one-to-one function. Hence, its inverse will not be a function.
Also, accept reasoning that since a horizontal line crosses the graph of f at more than one point then the graph of the inverse which is a reflection of f about the y-axis will have a vertical line crossing at more than one point indicating that one value in the domain (x) produces more than one value in the range (y). Hence, inverse of f is not a function.

(ii) The domain of f needs to be restricted so that g is a one-to-one function. By inspecting the graph of f, it can be deduced that the largest possible domain of g is $[-1,3]$.

(iii)



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(c) (i) $h(x)=\frac{2x-5}{x+d} \Rightarrow y=\frac{2x-5}{x+d}$

Switch domain and range, and solve for y:

$x=\frac{2y-5}{y+d} \Rightarrow xy+dx=2y-5 \Rightarrow xy-2y=-dx-5 \Rightarrow y(x-2)=-dx-5$

Thus, $h^{-1}(x)=\frac{-dx-5}{x-2}$

(ii) $h(x)=h^{-1}(x) \Rightarrow \frac{2x-5}{x+d}=\frac{-dx-5}{x-2}$; Thus, $d=-2$

(iii) $(h\circ k)(x)=h(k(x))=\frac{2k(x)-5}{k(x)-2}=\frac{2x}{x+1} \Rightarrow 2x\cdot k(x)-4x=2x\cdot k(x)-5x+2\cdot k(x)-5$

$2\cdot k(x)=x+5 \Rightarrow k(x)=\frac{x+5}{2}$

(d) $r(x)=\frac{ax+b}{cx+d} \Rightarrow y=\frac{ax+b}{cx+d}$

$x=\frac{ay+b}{cy+d} \Rightarrow cxy+dx=ay+b \Rightarrow cxy-ay=-dx+b \Rightarrow y(cx-a)=-dx+b$

$y=\frac{-dx+b}{cx-a} \Rightarrow r^{-1}(x)=\frac{-dx+b}{cx-a}$

In order for $r(x)=r^{-1}(x)$ then it must be that $\frac{ax+b}{cx+d}=\frac{-dx+b}{cx-a}$

Therefore, a function r in the form $r(x)=\frac{ax+b}{cx+d}$ is self-inverse if $a=-d$

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2. [Maximum mark: 30]

(a) $k=0$: curve $y = xe^x$ and line $y=0$ (x-axis)

intersection: $xe^x = 0 \Rightarrow x=0$ or $e^x = 0$

$e^x > 0,\ x \in \mathbb{R}$, therefore $y = xe^x$ and $y=0$ intersect only one when $x=0$ and $y=0$ (at the origin)

(b) $k=1$: line is $y=x$

find equation of line tangent to $y = xe^x$ at $(0,\ 0)$

$\frac{dy}{dx} = xe^x + e^x$; at $(0,\ 0)$: $\frac{dy}{dx} = 0 + e^0 = 1$

equation of tangent line is $y - 0 = 1\cdot(x-0) \Rightarrow y = x$ **Q.E.D.**

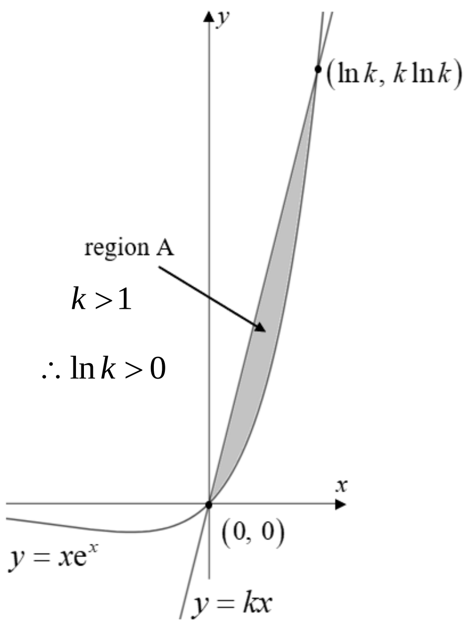
(c) (i) $xe^x = kx \Rightarrow x(e^x - k) = 0 \Rightarrow x=0$ or $x = \ln k$

$\ln k$ exists when $k>0$; however, when $k=1,\ x=\ln 1 = 0$ and there are not two distinct points of intersection
Therefore, there are two distinct points of intersection when $k>0,\ k\neq 1$

(ii) $xe^x = kx \Rightarrow x(e^x - k) = 0 \Rightarrow x=0$ or $x = \ln k$

when $x=0,\ y=0$; when $x=\ln k,\ y = k\ln k$

coordinates of points of intersection are $(0,\ 0)$ and $(\ln k,\ k\ln k)$



(d) (i) area of $A = \int_0^{\ln k} (kx - xe^x)\,dx$

(ii) $k = e^2$: area of $A = \int_0^{\ln(e^2)} (e^2 x - xe^x)\,dx$

area of $A = \int_0^2 e^2 x\,dx - \int_0^2 xe^x\,dx$

$$= \left[\frac{e^2x^2}{2}\right]_0^2 - \int_0^2 xe^x\,dx$$

Find $\int xe^x\,dx$ by integration by parts:

$u = x \Rightarrow du = dx$; $dv = e^x dx \Rightarrow v = e^x$

$$\int xe^x\,dx = xe^x - \int e^x\,dx$$
$$= xe^x - e^x$$

area of $A = \left[\frac{e^2x^2}{2}\right]_0^2 - \left[xe^x - e^x\right]_0^2$

$$= \left[2e^2 - 0\right] - \left[(2e^2 - e^2) - (0-1)\right]$$
$$= 2e^2 - 2e^2 + e^2 - 1$$

Thus, when $k = e^2$, area of $A = e^2 - 1$

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(iii) $k = \mathrm{e}^n,\ n \in \mathbb{R}^+$

area of $\mathrm{A} = \int_0^{\ln(\mathrm{e}^n)} \left(\mathrm{e}^n x - x\mathrm{e}^x\right)\mathrm{d}x$

$$= \left[\frac{\mathrm{e}^n x^2}{2} - \left(x\mathrm{e}^x - \mathrm{e}^x\right)\right]_0^n$$

$$= \left(\frac{n^2}{2}\mathrm{e}^n - n\mathrm{e}^n + \mathrm{e}^n\right) - (0 - 0 + 1)$$

Thus, when $k = \mathrm{e}^n,\ n \in \mathbb{R}^+$, area of $\mathrm{A} = \mathrm{e}^n\left(\frac{n^2}{2} - n + 1\right) - 1$ ***Q.E.D.***

(e) (i) $y = x\mathrm{e}^x \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = x\mathrm{e}^x + \mathrm{e}^x = \mathrm{e}^x(x+1) \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 0$ at $x = -1$

$y(-1) = -\mathrm{e}^{-1} = -\frac{1}{\mathrm{e}}$; therefore, coordinates of P are $\left(-1, -\frac{1}{\mathrm{e}}\right)$

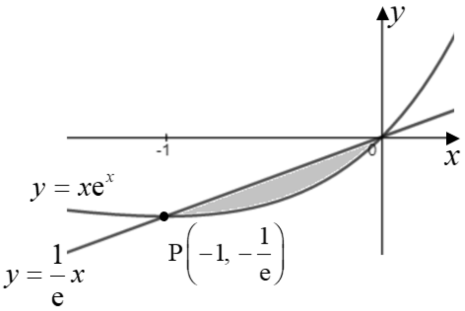
gradient of line $= k = \dfrac{0 - \left(-\frac{1}{\mathrm{e}}\right)}{0 - (-1)} = \frac{1}{\mathrm{e}} \Rightarrow k = \frac{1}{\mathrm{e}}$

(ii) $k = \frac{1}{\mathrm{e}}$: area of enclosed region $= \int_{-1}^{0}\left(\frac{1}{\mathrm{e}}x - x\mathrm{e}^x\right)\mathrm{d}x$

$$= \left[\frac{\mathrm{e}^{-1}x^2}{2} - \left(x\mathrm{e}^x - \mathrm{e}^x\right)\right]_{-1}^{0}$$

$$= (0 - 0 + 1) - \left(\frac{1}{2\mathrm{e}} + \frac{1}{\mathrm{e}} + \frac{1}{\mathrm{e}}\right) = 1 - \left(\frac{1}{2\mathrm{e}} + \frac{2}{2\mathrm{e}} + \frac{2}{2\mathrm{e}}\right)$$

area $= 1 - \frac{5}{2\mathrm{e}}$



(f) since $0 < k < 1$, then $\ln k < 0$ and $x = \ln k$ is lower limit of integration

area of $\mathrm{B} = \int_{\ln k}^{0}\left(kx - x\mathrm{e}^x\right)\mathrm{d}x$

$$\mathrm{B} = \left[\frac{k}{2}x^2 - \left(x\mathrm{e}^x - \mathrm{e}^x\right)\right]_{\ln k}^{0} = 0 - 0 + \mathrm{e}^0 - \left(\frac{k}{2}(\ln k)^2 - \ln k\left(\mathrm{e}^{\ln k}\right) + \mathrm{e}^{\ln k}\right)$$

$$= 1 - \frac{k}{2}(\ln k)^2 - k\ln k + k$$

$$= 1 - \frac{k}{2}\left[(\ln k)^2 - 2\ln k + 2\right]$$

$$= 1 - \frac{k}{2}\left[(\ln k)^2 - 2\ln k + 1 + 1\right] \qquad \left[(\ln k - 1)^2 = (\ln k)^2 - 2\ln k + 1\right]$$

$\mathrm{B} = 1 - \frac{k}{2}\left[(\ln k - 1)^2 + 1\right]$; $k > 0$ and $(\ln k - 1)^2 + 1 > 0$, therefore $\frac{k}{2}\left[(\ln k - 1)^2 + 1\right] > 0$

Thus, $\mathrm{B} = 1 - \frac{k}{2}\left[(\ln k - 1)^2 + 1\right] < 1$ ***Q.E.D.***